Do dissipative weak Euler solutions dream of turbulence?

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This title is a parody of a seminal science fiction novel by P.K. Dick [1]. The novel's subject is a challenge to what it is to be a human. Here we would like to consider the following questions: What is the essence of fluid turbulence in the sense of physics or mathematics? What does it mean to be a turbulent flow?

Obviously, we know that the Navier-Stokes (NS) equations have numerical solutions which simulate quite well turbulent flows in real situations. We also know a number of important laws on certain averages of the turbulent velocity obtained in laboratory experiments or numerical simulations of the NS equations. By laws, we mean that the statistical quantity always behaves the same way and does not depend on details of the system setting. In other words, such laws are observed in the flows regardless it is a part of a jet or wake of some obstacle etc. The famous laws include the Kolmogorov's $-5/3$ energy spectrum and the logarithmic law of the wall. Furthermore these universal nature of these laws are believed to originate from the nonlinearity of the fluids when it sufficiently dominates over other effects.

One crucial aspect of the two famous laws is that the range of spatial scales where they hold (known as the inertial range or the logarithmic region) extends as we decrease the kinematic viscosity, ν. We believe heuristically that the inertial range becomes infinitely long as $\nu \to 0$. This is a singular limit so that classical solutions of the Euler equations (i.e., $\nu = 0$) behave quite differently. For example, numerical solutions of the Euler equations do not produce the Kolmogorov's $-5/3$ law of the energy spectrum, $E(k) \propto \epsilon^{2/3}k^{-5/3}$, where ϵ is the energy dissipation rate. In the inviscid limit $\nu \to 0$, it is assumed that the energy dissipation rate tends to a constant which is independent of ν. This is the central hypothesis on turbulent flows in three dimensions. Basically, we need dissipation for the law to hold even though we consider the limit of vanishing dissipation.

In the end of 1940's, a visionary physicist L. Onsager stated that the energy can dissipate in the inviscid case $\nu = 0$, if the velocity is sufficiently rough. More specifically, he conjectured that the energy dissipation cannot occur if the velocity is smooth enough: $|\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t) - \boldsymbol{u}(\boldsymbol{x},t)| \le |\boldsymbol{r}|^h$ for $h < 1/3$ [2] (see also [7]). This critical exponent $1/3$ is the same one if we invoke the Kolmogorov dimensional analysis leading to the $-5/3$ law. The modern formulation of the conjecture in terms of the weak solution of the Euler equations was done in [3]. Then The proof was given in [3, 4]. The weak solutions of the Euler equations which dissipate the energy are called dissipative weak solutions.

An important development was later made in [5, 6] showing that certain dissipative weak solutions, if they exist, follow the Kolmogorov's $4/5$ law, as the NS turbulence do. This $4/5$ law is the most significant statistical law of turbulence since this is the only law that can be derived theoretically from the NS equations. However, between the dissipative weak solutions and the (classical) NS solutions, there is a huge difference in the condition of the $4/5$ law. For the former, the $4/5$ law holds for each solution, that is, without taking an ensemble average. For the latter, an ensemble average is indispensable. What does this imply? Our interpretation is that such a

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dissipative weak solution is an idealized form of turbulent flow. We here consider that something satisfies the 4/5 law has certain essence of turbulence.

There is a tradition of constructing weak solution of the Euler equations. Scheffer's [8] and Schnirelman's [9] solutions are famous examples. The dissipative weak solutions we talked about were constructed by De Lellis and Székelyhidi [10]. Specifically, they first prescribed the energy as a function of time and constructed iteratively a weak solution in agreement with the prescribed energy function. In [10], the Hölder exponent of the velocity was $h < 1/10$. Substantial efforts were then made to increase the exponent to the Onsager critical value 1/3. We just mention here that Isset [11] and Buckmaster *et al.* [12] reached the exponent arbitrary close to 1/3 from below. If one believes that certain dissipative weak solutions are an idealization of turbulence, their constructed solutions are likely to yield a novel insight on the mechanism of laws of turbulence. To what extent these weak solutions are relevant in understanding real turbulent flow is not obvious a priori. Of course, even if it turns out that they are of little relevance, their mathematical ingenuity is not at all reduced.

The construction [12] allows to prescribe two things: the energy as a continuous function of time and the Hölder exponent h of the velocity field, $|\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r},t) - \boldsymbol{v}(\boldsymbol{x},t)| \leq |\boldsymbol{r}|^h$. This h can be arbitrary close to 1/3 from below. The prescription of the energy, which can increase or decrease in time, avoids a rather pathological weak solution which has a compact support in time. The prescription of the Hölder exponent is quite intriguing for physicists interested in turbulence.

The classical picture of turbulence due to Kolmogorov proposed in 1940's, the exponent of the velocity field is uniquely 1/3. On the contrary, experiments and numerical simulations of the NS equations indicate that such exponents are multiple and continuously distributed around 1/3, see, e.g., [13]. Furthermore, the multiple exponents of the turbulent velocity field can be related to inhomogeneous fluctuation of the energy dissipation rate. This fluctuation means that in some points the energy dissipation rate can be enormously larger than the average. The Kolmogorov's refined theory of turbulence presented in 1960's took this sort of fluctuations into consideration and it predicted a certain distribution of the exponents of the turbulent velocity. Unfortunately, the refined theory is not able to describe well the real turbulent flow quantitatively. However the theoretical direction it opened up remains influential.

How is this multiplicity of the exponents or the peculiar fluctuation of the energy dissipation rate produced? This is one of the big questions in the physics of turbulence. In particular, since the energy dissipation rate has something to do with drag in real applications, answering the question may have some contribution in engineering area. Does the construction [12] provide any insight in the question? We believe so. This is why we initiated a numerical implementation of the construction. If the solution has the unique exponent β as prescribed, then we can learn why it remains unique. If the solution has multiple exponents, then we must learn why they are so and, in particular, what element in the construction determines the distribution of the exponents.

In fact, when we consider relevance of the weak solutions to the real turbulent flow, what physicists and engineers believe to know about it is challenged somehow. Said differently, they are asked to formulate their knowledge, which is sometimes quite fuzzy, in a more precise way and redefine it. This is perhaps the most interesting thing occurring now in this field.

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References

[1] P.K. Dick, *Do androids dream of electric sheep?*, Doubleday (1968).

[2] L. Onsager, Nouvo Cimento Suppl., **6**, 287–297 (1949).

[3] G.L. Eyink, Physica D, **78** 222–240 (1994).

[4] P. Constantin, W. E and E. Titi, Comm. Math. Phys. **165**, 207–209 (1994).

[5] J. Duchon and R. Robert, Nonlinearity **13**, 249–255 (2000).

[6] G.L. Eyink, Nonlinearity **16**, 137–146 (2002).

[7] G.L. Eyink and K.R. Sreenivasan, Rev. Mod. Phys., **78**, 87–135 (2006).

[8] V. Scheffer, J. Geom. Anal., **3**, 343–401 (1993).

[9] A.I. Shnirelman, Comm. Pure Appl. Math., **50** 1261–1286 (1997).

[10] C. De Lellis, and L. Székelyhidi Jr., Invent. Math. 193–377 (2013)

[11] P. Isset, arXiv:1608:08301, (2016).

[12] T. Buckmaster, C. De Lellis, L. Székelyhidi Jr. and V. Vcol, arXiv:1701.08678 (2017).

[13] U. Frisch, *Turbulence*, Cambridge univ. press (1995).